

Spectra of Products and Numerical Ranges¹

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1. INTRODUCTION

If A is bounded linear transformation from a complex Hilbert space H into itself, then the *numerical range* of A is by definition the set

$$W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}.$$

It is wellknown and easy to prove that if $\sigma(A)$ denotes the spectrum of A , then

$$\sigma(A) \subset \overline{W(A)},$$

where the bar indicates closure.

The purpose of this paper is two-fold. We first present an extension of the foregoing relation and the proceed to indicate how the extension may be used in two other situations, namely bounded linear operators on a Banach space, and certain nonlinear transformations on a real or complex Hilbert space. The extension is mild, Specifically, we will show that if $0 \notin \overline{W(A)}$, then

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$$

for any operator B on H . Here the set on the right is by definition the set of quotients b/a with $b \in \overline{W(B)}$ and $a \in \overline{W(A)}$.

The extension has interesting consequences. For example it implies that if A is strictly positive and $B \geq 0$, then the product AB has a nonnegative spectrum. Also, if A is positive and B is self-adjoint then the product AB has real spectrum.

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2. LINEAR OPERATORS ON A HILBERT SPACE

We begin with the proof of the extension.

THEOREM 1. *Let A and B operators on the complex Hilbert space H . If $0 \notin \overline{W(A)}$ then*

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}.$$

PROOF. Observe first of all that since $\sigma(A) \subset \overline{W(A)}$, the hypothesis guarantees that A^{-1} exists (as a bounded linear operator on H). Secondly, the identity

$$A^{-1}B - \lambda = A^{-1}(B - \lambda A)$$

shows that if $\lambda \in \sigma(A^{-1}B)$, then $0 \in \sigma(B - \lambda A)$. This in turn implies that

$$0 \in \overline{W(B - \lambda A)} \subset \overline{W(B)} - \lambda \overline{W(A)},$$

and this means that

$$\lambda \in \overline{W(A)}/\overline{W(A)}.$$

We indicated two corollaries above. To get another we recall that any operator A on H has a "polar decomposition"

$$A = UP,$$

and that if A is invertible, then U is unitary and P is strictly positive. Following Berberian [1] we call the unitary operator U *cramped* if its spectrum is contained in an arc of the unit circle with central angle $< \pi$.

COROLLARY (Berberian). *If $0 \notin \overline{W(A)}$, then the unitary part of A is cramped.*

PROOF. Use the fact that $\overline{W(A)}$ is convex to see that if $0 \notin \overline{W(A)}$, then $\overline{W(A)}$ is contained in a sector

$$S = \{re^{i\theta} : r > 0 : \theta_1 \leq \theta \leq \theta_2\}$$

with $\theta_2 - \theta_1 < \pi$. Then write $U = A \cdot P^{-1}$ and apply the theorem to see that $\sigma(U)$ is a subset of the arc

$$\{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}.$$

REMARK. (i) The inclusion $\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$ is not valid with the

weaker assumption that A is merely invertible. Indeed if A and B are self-adjoint $\sigma(AB)$ need not even be real. This follows from the computation

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in two-dimensional Hilbert space

(ii). The more symmetric statement

$$\sigma(AB) \subset \overline{W(A)} \cdot \overline{W(B)} \quad \text{if} \quad 0 \notin \overline{W(A)} \cup \overline{W(B)}$$

is also not valid. To see this let A be the operator

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $W(A) = W(A^*)$ is the disk of radius $1/2$ about 1 and so the set $W(A) \cdot W(A^*)$ lies to the left of $\operatorname{Re} z = 9/4$. On the other hand $9/4 < 1/2(3 + \sqrt{5}) \in \sigma(AA^*)$.

Returning to the theorem, the reader will note that the proof really does not concern operators on a Hilbert space at all. Indeed, the essential ingredients are these: An algebra \mathcal{A} with unit, and two mappings $A \rightarrow \sigma(A)$, $A \rightarrow W(A)$ from \mathcal{A} to subsets of the complex plane which have the following properties:

- (1) $W(A + B) \subset W(A) + W(B)$
- (2) $W(\lambda A) \subset \lambda W(A)$
- (3) $\sigma(A) \subset \overline{W(A)}$
- (4) $\lambda \notin \sigma(A)$ if and only if $(A - \lambda)^{-1} \in \mathcal{A}$.

(We write $B^{-1} \in \mathcal{A}$ to mean that the element B of \mathcal{A} has an inverse and that this inverse in fact belongs to \mathcal{A} .) In what follows we will indicate how this observation extends the theorem to two other situations.

3. LINEAR OPERATORS ON A BANACH SPACE

For our first application we need a few facts about Banach spaces. First, if X is a Banach space then the Hahn-Banach theorem guarantees that for each $x \in X$ there is an $x^* \in X^*$ of norm 1 such that $\langle x, x^* \rangle = \|x\|$. The space X (or more properly, the unit ball of X) is called *smooth* [2] if there is exactly one such x^* for each $x \in X$. Thus in a smooth space there is a unique map φ from X to X^* such that

$$\|\varphi(x)\| = \|x\|, \quad \langle x, \varphi(x) \rangle = \|x\|^2 \quad (x \in X).$$

As an example the reader can easily verify that L^p is smooth for $1 < p < \infty$. The isometry φ sends $f \in L^p$ to

$$f \frac{|f|^{p-2}}{\|f\|^{p-2}}.$$

If X is smooth and φ is the indicated mapping, then it is easy to see that φ is conjugate homogeneous:

$$\varphi(\alpha x) = \bar{\alpha}\varphi(x), \quad \alpha \text{ complex.}$$

(However, if φ is additive, then the norm in X satisfies the parallelogram law and hence X is a Hilbert space.) Again, if X is smooth and $f \in X^*$ attains its supremum on the unit ball of X , then f belongs to the range of φ . Now a result of Bishop and Phelps [3] states that for any Banach space X the collection of bounded linear functionals on X which attain their suprema on the unit ball of X is always (norm) dense in X^* . By using this fact and the preceding remark it follows that if X is smooth, then the range of φ is dense in X^* .

Now using the function φ we can define a "semi-inner-product" on X by

$$[x, y] = \langle x, \varphi(y) \rangle \quad (x, y \in X).$$

It is readily verified that the following hold:

$$\begin{aligned} [x, x] &= \|x\|^2 \\ [x_1 + x_2, y] &= [x_1, y] + [x_2, y] \\ [\lambda x, y] &= \lambda[x, y], [x, \lambda y] = \bar{\lambda}[x, y] \\ |[x, y]| &\leq \|x\| \|y\|. \end{aligned}$$

If now A is a bounded linear operator on X we can define the *numerical range* of A by setting

$$W(A) = \{[Ax, x] : \|x\| = 1\}.$$

Clearly we will have

$$\begin{aligned} W(A + B) &\subset W(A) + W(B), \\ W(\lambda A) &\subset \lambda W(A). \end{aligned}$$

Lumer [4] also shows that the boundary of $\sigma(A)$ is a subset of $\overline{W(A)}$. We need the following stronger result:

PROPOSITION. $\sigma(A) \subset \overline{W(A)}$.

PROOF. The argument parallels the linear case: If λ is at a positive distance δ from $\overline{W(A)}$, then for unit vectors x

$$\|(A - \lambda)x\| \geq |[A - \lambda)x, x]| = |[Ax, x] - \lambda| \geq \delta = \delta \|x\|$$

and

$$\|(A - \lambda)^* \varphi(x)\| \geq |\langle x, (A - \lambda)^* \varphi(x) \rangle| = \|[(A - \lambda)x, x]\| \geq \delta = \delta \|\varphi(x)\|.$$

The first of these implies that $A - \lambda$ is one-to-one with a closed range. The second implies that $(A - \lambda)^*$ is bounded below on the range of φ and since this is dense in X^* , $(A - \lambda)^*$ is bounded below, hence one-to-one, and this means that $A - \lambda$ has a dense range. It now follows from the Open Mapping Theorem that $A - \lambda$ has a bounded inverse. Hence $\lambda \notin \overline{W(A)}$ implies $\lambda \notin \sigma(A)$ as asserted.

We may summarize the preceding discussion as follows:

THEOREM 2. *Let X be a smooth Banach space and define $W(A)$ as above. Then if $0 \notin \overline{W(A)}$ we have*

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$$

for any operator B on X .

If the Banach space X is not smooth then there will be many isometries φ_α from X to X^* satisfying

$$\langle x, \varphi_\alpha(x) \rangle = \|x\|^2 \quad (x \in X).$$

Each of these maps defines a semi-inner product $[\ , \]_\alpha$ on X and a bounded linear operator T on X has corresponding numerical ranges $W_\alpha(T)$. It is natural to define the *numerical range* of T on X by

$$W(T) = \bigcup_{\alpha} W_\alpha(T).$$

The argument used for the smooth case is easily adapted to prove that $\sigma(T) \subset \overline{W(T)}$ is still valid and so we can conclude that Theorem 2 holds without the hypothesis that X is smooth.

In this connection Lumer has shown [4] that $W(T)$ is real (or positive) if and only if some $W_\alpha(T)$ is real (or positive). Thus $T = T^*$ (or $T \geq 0$) has intrinsic meaning and with these conventions we can state the following corollary:

COROLLARY. *If $A > 0$, $B \geq 0$ and $C = C^*$, then $\sigma(AB)$ is positive and $\sigma(AC)$ is real.*

4. NONLINEAR OPERATORS ON A HILBERT SPACE

Our final application is more delicate. Here we let H be a real or complex Hilbert space and let \mathcal{A} be the collection of maps from H to itself which are

continuous and which send bounded sets into bounded sets. Clearly \mathcal{A} is an algebra with unit. We take the numerical range of $A \in \mathcal{A}$ to be

$$W(A) = \left\{ \frac{\langle Ax_1 - Ax_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|^2} : x_1 \neq x_2 \right\}.$$

There are two possible definitions of the spectrum of $A \in \mathcal{A}$, namely, $\sigma(A)$, and $\sigma_1(A)$ defined respectively as the complements of the sets

$$\rho(A) = \{\lambda : (A - \lambda)^{-1} \in \mathcal{A}\}$$

$$\rho_1(A) = \{\lambda : (A - \lambda)^{-1} \text{ exists and is Lipschitzian}\}.$$

(By definition, B is Lipschitzian if

$$\|Bx_1 - Bx_2\| \leq M \cdot \|x_1 - x_2\|$$

for some constant $M > 0$ and all x_1, x_2 .)

It is easy to see that $\sigma(A) \subset \sigma_1(A)$. Moreover, a theorem of Zarantonello [5] asserts that, with $W(A)$ as defined above, we have the inclusion

$$\sigma_1(A) \subset \overline{W(A)}.$$

Taking $\sigma(A)$ as the definition of the spectrum of A and applying Theorem 1, we get the following result:

THEOREM 3. *Let A and B be bounded and continuous on H . If $0 \notin \overline{W(A)}$, then for each $\lambda \notin \overline{W(B)} \setminus \overline{W(A)}$ the mapping $A^{-1}B - \lambda$ has a bounded, continuous inverse defined on H .*

Taking $\sigma_1(A)$ as the definition of the spectrum of A we get:

THEOREM 4. *Let B be bounded and continuous, let A be Lipschitzian and suppose $0 \notin \overline{W(A)}$. Then for each λ outside the set $\overline{W(B)} \setminus \overline{W(A)}$ the transformation $A^{-1}B - \lambda$ has a Lipschitzian inverse defined on H .*

PROOF. If $0 \notin \sigma_1(B - \lambda A)$, then $(B - \lambda A)^{-1}$ exists and is Lipschitzian. Hence the product $(B - \lambda A)^{-1}A$ is also Lipschitzian. Since however

$$(A^{-1}B - \lambda)(B - \lambda A)^{-1}A = A^{-1}(B - \lambda A)(B - \lambda A)^{-1}A = 1,$$

this implies that $A^{-1}B - \lambda$ has a Lipschitzian inverse and so $\lambda \notin \sigma_1(A^{-1}B)$. In other words,

$$\lambda \in \sigma_1(A^{-1}B) \Rightarrow 0 \in \sigma_1(B - \lambda A)$$

and the remainder of the proof is as before.

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